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# Infinitely Many Contact Process Transitions on a Tree

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We continue our study of the ergodic behavior of the contact process on infinite connected graphs of bounded degree. Examples are provided of trees on which, as the infection parameter increases, complete convergence alternates between holding and failing infinitely many times.

**KEY WORDS:** Contact process; graphs; invariant measures; ergodic behavior; critical points; complete convergence; partial convergence.

## 1. INTRODUCTION

This is a continuation of a study of the behavior of the contact process on general graphs of bounded degree, initiated in Salzano and Schonmann (1997) and (1999). The contact process is one of the basic interacting particle systems [see Liggett (1985), Liggett (1999), or Durrett (1988)]. It can be described as follows. At each vertex (site) of a graph $G$ there is an individual which may be healthy or infected with some disease. As time evolves, infected individuals may recover at rate one, independently of anything else, but each infected individual also infects its neighbors at rate $\lambda > 0$ independently of anything else.

The contact process is said to survive globally if starting from finitely many infected vertices, there is a positive probability that the infection will never disappear from the graph. It is said to survive locally (or recur) if starting from finitely many infected vertices there is a positive probability that the infection will recur forever to any vertex of the graph.

A particularly nice behavior that the contact process may have for some values of $\lambda$ is called complete convergence (cc). This concept will be defined precisely in the next section, but roughly it means that there are

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only two extremal invariant probability distributions and starting from any state there is convergence to a certain combination of these two.

Pemantle (1992) conjectured that on any graph once the contact process survived locally, it would satisfy cc. For the cubic lattices $\mathbb{Z}^d$ and the homogeneous trees $\mathbb{T}^d$ this is true [see respectively Bezuidenhout and Grimmett (1990), Zhang (1996) or Salzano and Schonmann (1998)]. But Salzano and Schonmann (1997) disproved this conjecture and provided examples of trees with bounded degree on which, as $\lambda$ increases, cc alternates between holding and failing any finite number of times. Here we will show that the behavior on a graph of bounded degree can be even more complex, with infinitely many transitions in behavior occuring as $\lambda$ increases to $\lambda_c(\mathbb{Z})$, the critical parameter for the contact on the lattice $\mathbb{Z}$. [It has been proven in Salzano and Schonmann (1997) that on any infinite graph of bounded degree, when $\lambda > \lambda_c(\mathbb{Z})$, complete convergence holds.]

## 2. NOTATION AND BACKGROUND

This section can be seen as a summary. Readers who need more details are invited to read the introduction of Salzano and Schonmann (1997), and consult Liggett (1985) and Durrett (1988).

### 2.1. The Graphs

We will denote by $\mathcal{G}$ the class of infinite connected graphs of bounded degree. Given $G = (\mathcal{V}_G, \mathcal{E}_G) \in \mathcal{G}$, the notation $A \Subset \mathcal{V}_G$ stands for $A$ being a finite subset of the set of vertices of $G$, $\mathcal{V}_G$. An arbitrary vertex $0 \in \mathcal{V}_G$ is called the root of $G$. We will measure the distance between sites in $\mathcal{V}_G$, $G \in \mathcal{G}$, by the length of the minimal path along neighboring sites which joins them. The degree of a vertex $x \in \mathcal{V}_G$ is the number of edges connected to it.

### 2.2. The Contact Process

The contact process with infection parameter $\lambda$ started from the configuration $A$ is denoted by $(\xi^A_{G;\lambda;t})_{t \geqslant 0}$. The probability of global survival is denoted by $\rho_G(A, \lambda) = \mathbb{P}\{\xi^A_{G;\lambda;t} \neq \emptyset$, for all $t \geqslant 0\}$. Similarly the probability of local survival or recurrence is denoted by $\beta_G(A, \lambda) = \mathbb{P}\{\xi^A_{G;\lambda;t}(0) = 1$, infinitely often$\}$. In typical abuses of notation $G$, $A$ or $\lambda$ may be omitted from the notation; in the case of $A$ meaning $A = \{0\}$.

Next we define the corresponding critical points for $G$:

$$\lambda_s = \inf\{\lambda : \rho(\lambda) > 0\} \qquad \text{and} \qquad \lambda_r = \inf\{\lambda : \beta(\lambda) > 0\}$$

These are, respectively, called the survival point and the recurrence point of contact process on $G$.

## 2.3. The Ergodic Behavior of the Contact Process

The lower and the upper invariant measures of the contact process are respectively $\delta_{\varnothing}$, the point mass in the empty configuration, and $\bar{\nu}$, the invariant measure approached as time goes to infinite when the process starts with all individuals infected. A classical notion of convergence in distribution in the study of the contact process is the following:

**Complete Convergence (cc).** For any $A \Subset \mathscr{V}_G$, $\xi_t^A \Rightarrow (1-\rho(A))\,\delta_{\varnothing} + \rho(A)\,\bar{\nu}$, as $t \to \infty$.

Before we can summarize some results of Salzano and Schonmann (1997) and (1999) to be used here, we need to review the following terminology:

The self-duality of the contact process implies that $\bar{\nu}(\zeta : \zeta \cap A \neq \varnothing) = \rho(A)$. Motivated by this, we introduced in Salzano and Schonmann (1997) the probability distribution $\nu_r$, defined by: $\nu_r(\zeta : \zeta \cap A \neq \varnothing) = \mathbb{P}(\xi_t^A(0) = 1$, infinitely often$) = \beta(A)$. We proved there that $\nu_r$ is an extremal invariant measure. We also defined:

**Criterion r=s.** $\nu_r = \bar{\nu}$. Equivalently, for all non-empty $A \Subset \mathscr{V}_G$, $\beta(A) = \rho(A)$.

Motivated by the definition of cc, we introduced in Salzano and Schonmann (1997) the following similar notion:

**Partial Convergence (pc).** For any $A \Subset \mathscr{V}_G$, $\xi_t^A \Rightarrow (1-\beta(A))\,\delta_{\varnothing} + \beta(A)\,\nu_r$, as $t \to \infty$.

We define by s&cc (for survival with complete convergence) the property that $\rho(\lambda) > 0$ and cc holds. Similarly, we define r&pc (for "recurrence with partial convergence") as the property that pc holds and $\beta(\lambda) > 0$.

**Monotone Increasing Property.** A property of the contact process is said to be monotone increasing when both of the following hold.

(i) If the property holds for the contact process on a graph $G \in \mathscr{G}$ at some $\lambda$, then it also holds for the same graph for all $\lambda' > \lambda$.

(ii) If the property holds for the contact process on some subgraph $G_0 \in \mathcal{G}$ of some graph $G \in \mathcal{G}$ at some value of $\lambda$, then it also holds for $G$ at the same $\lambda$.

**Collage of Graphs.** Suppose that $G_1,..., G_n \in \mathcal{G}$ are disjoint graphs. We say that $G \in \mathcal{G}$ is a collage of $G_1,..., G_n$ if the following conditions are satisfied:

(i) The set of vertices of $G$ is $\mathcal{V}_G = (\bigcup_{i=1,...,n} \mathcal{V}_{G_i}) \cup V_0$, where $V_0$ is a finite set.

(ii) The set of edges of $G$ is $(\bigcup_{i=1}^{n} \mathcal{E}_{G_i}) \cup E_0$, where $E_0$ is a finite set.

The statements (A) and (B) below are part of Theorem 2 of Salzano and Schonmann (1997). The statement (C) is Theorem 2.1.1 of Salzano and Schonmann (1999), but its proof can be found in a less general form in Salzano and Schonmann (1997).

(A) The property r&pc is monotone increasing.

(B) Having cc is equivalent to having simultaneously both pc and r = s.

(C) Suppose that $G$ is a collage of $G_1,..., G_n$, then for each value of $\lambda > 0$ the condition r = s holds for $G$ if and only if it holds for each one of the graphs $G_1,..., G_n$.

## 3. RESULTS

We will give act example of a tree in $\mathcal{G}$ on which, as $\lambda$ increases, cc alternates between holding and failing infinitely many times.

For each positive integer $l$, define $\mathbb{T}_{2,l}$ as the tree given by the rule that for $n = 0, 1, 2,...$ the vertices at distance $nl$ from the origin have degree three and all the other vertices have degree two. Notice that the tree $\mathbb{T}_{2,1}$ is the same as the homogeneous tree $\mathbb{T}_2$. To define $\mathbb{T}_{2,l}^{+}$, remove one edge connected to the origin of $\mathbb{T}_{2,l}$ and call $\mathbb{T}_{2,l}^{+}$ the connected component containing the origin.

For any positive integer $l$, the tree $\mathbb{T}_{2,l}$ has:

$$0 < \lambda_s(\mathbb{T}_{2,l}) < \lambda_r(\mathbb{T}_{2,l}) < \lambda_c(\mathbb{Z}) \tag{3.1}$$

The second inequality comes from Stacey (1996). The third one comes from Aizenman and Grimmett (1991) once we observe that $\mathbb{T}_{2,l}$ has a subgraph, $\mathbb{Z}^{(l)}$, constructed as follows. Label each vertex of $\mathbb{Z}$ according to its distance to the origin and for each integer $k$ add a new vertex and a new edge connecting it to the vertex $kl$ in $\mathbb{Z}$. The same arguments used in the proof of Theorem 4 of Salzano and Schonmann (1997) show that:

$$\lambda_s(\mathbb{T}^+_{2,l}) = \lambda_s(\mathbb{T}_{2,l}) \qquad \text{and} \qquad \lambda_r(\mathbb{T}_{2,l}) = \lambda_r(\mathbb{T}_{2,l})$$

Also arguments similar to those used for the homogeneous trees [see Morrow, Schinazi and Zhang (1994) and Zhang (1996) or Salzano and Schonmann (1998)] give

$$\rho_{\mathbb{T}^+_{2,l}}(\lambda_s(\mathbb{T}_{2,l})) = 0 \qquad \text{and} \qquad \beta_{\mathbb{T}^+_{2,l}}(\lambda_r(\mathbb{T}_{2,l})) = 0$$

To construct the tree with infinitely many transitions, we will choose a suitable sequence $\{l_i\}_{i=1,2,\dots}$. The tree $T(l_1, l_2,\dots)$ is then constructed as follows. Start with $\mathbb{Z}^+$, label each of its sites according to its distance to the origin. For each positive $k$ add an edge connecting the site $\sum_{i=1}^{k} l_i$ to the root of a copy of $\mathbb{T}^+_{2,l_k}$.

Define $\mathcal{G}_l$ to be the set of all trees with vertices of degree either two or three, with no infinite chains of vertices of degree 2 and where the distance between any two vertices of degree three is at least $l$. Let $\tilde{\lambda}(l) = \inf\{\lambda_s(G) : G \in \mathcal{G}_l\}$. Clearly $\tilde{\lambda}(l) \leqslant \lambda_s(\mathbb{T}_{2,l})$ for any $l$, since $\mathbb{T}^+_{2,l} \in \mathcal{G}_l$.

The main technical task in this paper will be the proof of the following:

**Lemma 1.** $\lim_{l\to\infty} \tilde{\lambda}(l) = \lambda_c(\mathbb{Z})$.

This lemma, and (3.1) assures us that we can take $l_1 < l_2 < l_3 < \cdots$ such that the following is satisfied:

$$\lambda_s(\mathbb{T}_{2,l_1}) < \lambda_r(T_{2,l_1}) < \tilde{\lambda}(l_2) \leqslant \lambda_s(\mathbb{T}_{2,l_2}) < \lambda_r(\mathbb{T}_{2,l_2}) < \tilde{\lambda}(l_3) \leqslant \cdots < \lambda_c(\mathbb{Z}) \tag{3.2}$$

Of course, the choice of $\{l_i\}_{i=1,2,\dots}$ can be made so that $l_{i+1}$ is a multiple of $l_i$, for $i = 1, 2,\dots$.

**Theorem.** Suppose that $l_1, l_2,\dots$ are such that (3.2) is satisfied. Then for the contact process on $T(l_1, l_2,\dots)$ we have:

(a) cc does not hold in $(\lambda_s(\mathbb{T}_{2,l_j}), \lambda_r(\mathbb{T}_{2,l_j})]$, $j = 1, 2,\dots$

(b) s&cc holds in $(\lambda_r(\mathbb{T}_{2,l_j}), \tilde{\lambda}(l_{j+1}))$, $j = 1, 2,\dots$

Moreover, if the sequence $\{l_i\}_{i=1,2,\dots}$ is such that $l_{i+1}$ is a multiple of $l_i$, for $i = 1, 2,\dots$, then s&cc holds in $(\lambda_r(\mathbb{T}_{2,l_j}), \lambda_s(\mathbb{T}_{2,l_{j+1}})]$.

*Remark.* We conjecture, but have not been able to prove that $\tilde{\lambda}(l) = \lambda_s(\mathbb{T}_{2,l})$ for all positive integer $l$.

## 4. PROOFS

*Proof of the Theorem.* In order to prove (a), let $\lambda \in (\lambda_s(\mathbb{T}_{2,l_j}), \lambda_r(\mathbb{T}_{2,l_j})]$, for some $j$. In this interval we know that the contact process on $\mathbb{T}^+_{2,l_j}$ does not satisfy r = s. The tree $T(l_l, l_2,...)$ is a collage of $\mathbb{T}^+_{2,l_j}$ and another tree. Since the first one does not satisfies r = s, by (C), the same happens to $T(l_1, l_2,...)$. It follows from (B) that in this case $T(l_1, l_2,...)$ does not satisfy cc.

For (b), fix $\lambda \in (\lambda_r(\mathbb{T}_{2,l_j}), \tilde{\lambda}(l_{j+1}))$, for some $j$. Notice that $T(l_1, l_2,...)$ is a collage of $\mathbb{T}^+_{2,l_1},..., \mathbb{T}^+_{2,l_j}$ and $T(l_{j+1}, l_{j+2},...)$. For such values of $\lambda$ we have that the process on $T(l_{j+1}, l_{j+2},...)$ dies out since $T(l_{j+1}, l_{j+2},...) \in \mathcal{G}_{l_{j+1}}$ and $\lambda < \tilde{\lambda}(l_{j+1})$. The contact process on each $\mathbb{T}^+_{2,l_i}$, $i = 1,..., j$ satisfies s&cc since $\lambda > \lambda_r(\mathbb{T}_{2,l_i})$ (same proof as in Zhang (1996) or Salzano and Schonmann (1998)). The process on a collage of a finite number of trees where the process dies out in some of them and in the others satisfies s&cc, also satisfies s&cc. For this, combine (A) with (B) and (C). Thus, the contact process on $T(l_1, l_2,...)$ satisfies s&cc.

In the case $l_{i+1}$ is a multiple of $l_i$, for $i = 1, 2,...$ then $T(l_{j+1}, l_{j+2},...)$ is a subgraph of $\mathbb{T}_{2,l_{j+1}}$, and hence the contact process on $T(l_{j+1}, l_{j+2},...)$ dies out when $\lambda \leqslant \lambda_s(\mathbb{T}_{2,l_{j+1}})$. So, the proof of (b) also gives the final claim in the theorem. ∎

*Proof of Lemma 1.* Fix $\lambda < \lambda_c(\mathbb{Z})$ and let $G \in \mathcal{G}_l$. We want to show the extinction of the contact process on $G$ when $l$ is large.

We will compare the contact process $(\xi^0_t)_{t \geqslant 0}$ on $G$ to a process $(\tilde{\xi}^0_t)_{t \geqslant 0}$ where the states of $(\tilde{\xi}^0_t)_{t \geqslant 0}$ are finite collections of particles on $G$. Each particle has a type and there is no more than one particle of each type in each site.

Let $\mathcal{V}^{(3)}$ be the set of vertices of $G$ of degree three. Particles of each type have a site in $\mathcal{V}^{(3)}$ they call their home.

Particles of different type evolve as independent contact processes unless a particle tries to infect a site $y \in \mathcal{V}^{(3)}$ different from its home. In this case, a particle with a type still not used is created at $y$. At time 0 the process $(\tilde{\xi}^x_t)_{t \geqslant 0}$ has just one particle at $x$, of type 0 and it has $x$ as its home.

Note that

$$\xi^0_t \leqslant \tilde{\xi}^0_t, \text{ in the sense that: } \forall x \in \mathcal{V}_G, \ \forall t \geqslant 0$$

$$\xi^0_t(x) = 1 \Rightarrow \tilde{\xi}^0_t \text{ has at least one particle at } x$$

To each type of particle we associate a generation number. Particles of type 0 have generation number 0. When a particle of a type which is of generation $n$ creates a new type of particle, this new type of particle is said to be of generation $n+1$. Let $N_k$ be the number of types of particles of generation $k$ ever created.

Given $x \in \mathscr{V}^{(3)}$, let $\mu_x$ be the expected number of types of particles of generation 1 created in the process $(\tilde{\xi}_t^x)_{t \geqslant 0}$.

Set

$$\mu(G) = \sup_{x \in \mathscr{V}^{(3)}} \mu_x$$

Since it is clear that the multi-valued process $(\tilde{\xi}_t^x)_{t \geqslant 0}$ cannot survive forever restricted to a finite set of vertices, in the event of survival, infinitely many different types of particles must be created. But the next result will show that this is not the case when $\lambda < \lambda_c(\mathbb{Z})$.

Clearly

$$E(N_k \mid N_{k-1}) \leqslant \mu N_{k-1}$$

So, $E(N_k) \leqslant \mu^k$. Using the Lemma 2 below, we can take $l$ large enough so that $\mu < 1$. Therefore the number of types of particles ever created will be a.s. finite. ∎

In the proof of the lemmas below, we will use several well-known exponential estimates for the sub-critical contact process [see Liggett (1985)]. For later reference we recall them now. Suppose $\lambda < \lambda_c(\mathbb{Z})$, then for some positive finite constants $c_1$, and $c_2$,

(a) $P(\xi^0_{\mathbb{Z};\, t} \neq \varnothing) \leqslant c_1 \exp\{-c_2 t\}$.

(b) $P((\xi^0_{\mathbb{Z};\, t})_{t \geqslant 0}$ reaches the site $k) \leqslant c_1 \exp\{-c_2 k\}$.

(c) $P(\inf \xi^{\mathbb{Z}^+}_{\mathbb{Z};\, t} \leqslant \exp\{c_2 t\}$ for some $t \geqslant t_0) \leqslant c_1 \exp\{-c_2 t_0\}$.

**Lemma 2.** Suppose $\lambda < \lambda_c(\mathbb{Z})$. Then $\lim_{l \to \infty} \sup_{G \in \mathscr{G}_l} \mu(G) = 0$.

*Proof of Lemma 2.* Given $G \in \mathscr{G}_l$ and $x \in \mathscr{V}^{(3)}$, we define the star centered at $x$, $S_x$, as the maximal connected subgraph of $G$ which has $x$ as the only vertex of $\mathscr{V}^{(3)}$. The tips of $S_x$ are the vertices of this subgraph which have degree one in $S_x$. The set of tips of $S_x$ will be denoted by $\mathscr{T}_x$.

Note that, by conditioning on the time the sites in $\mathscr{T}_x$ are infected, one obtains

$$\mu_x = \lambda \sum_{y \in \mathscr{T}_x} \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\{\xi^x_{S_x;\, t}(y) = 1\}}\, dt\right) \tag{4.1}$$

Also, for $y \in \mathscr{T}_x$, using self-duality and (b),

$$\mathbb{E}\int_0^l \mathbb{1}_{\{\xi^x_{S_x;\,t}(y)=1\}}\,dt \leqslant \mathbb{E}\int_0^l \mathbb{1}_{\{(\xi^y_{S_x;\,s})_{s\geqslant 0}\text{ reaches }x\}}\,dt \leqslant (c_1 l)\exp\{-c_2 l\} \tag{4.2}$$

Let $S$ be the tree in which one vertex, the root, has degree three and all other vertices have degree two.

If we think of $S_x$ as a subgraph of $S$, Lemma 3 below gives:

$$\mathbb{E}\int_l^\infty \mathbb{1}_{\{\xi^x_{S_x;\,t}(y)=1\}}\,dt \leqslant \mathbb{E}\int_l^\infty \mathbb{1}_{\{\xi^S_{S;\,t}(y)=1\}}\,dt \leqslant \int_l^\infty c_3\exp\{-c_4 t/\log t\}\,dt \tag{4.3}$$

The lemma follows from (4.1), (4.2) and (4.3).

**Lemma 3.** Suppose $\lambda < \lambda_c(\mathbb{Z})$. Then there are positive finite constants $c_3$ and $c_4$ such that for every vertex $y$ of $S$ and $t \geqslant 0$,

$$P(\xi^S_{S;\,t}(y)=1) \leqslant c_3\exp\{-c_4 t/\log t\}$$

*Proof of Lemma 3.* Let $H_r$ be the set of the configurations in which all vertices of $S$ within distance $r$ of 0 are vacant.

Set

$$T = \sup\{t : \xi^S_{S;\,t} \notin H_t\}$$

We will first argue that:

$$P(T>t) \leqslant c_3\exp\{-c_4 t/\log t\} \tag{4.4}$$

Let $A_n$ be the event that if we start the contact process on $S$ at time $(n-1)\,C\log t$ from the configuration with all vertices occupied, then the process will be in the set $H_{2t}$ at time $nC\log t$. The events $A_1, A_2,...$ are independent and for large enough $C$, so we have

$$P(A_n) \geqslant \frac{1}{1+3\lambda}\,[P\{(\xi^{\mathbb{Z}^+}_{\mathbb{Z},\,t})_{t\geqslant 0}\text{ is contained in }\mathbb{Z}^+\text{ and at time }C\log t$$
$$\text{the sites }\{0, 1,..., \lfloor 2t\rfloor\}\text{ are empty}\}]^3 \geqslant \varepsilon$$

where $\varepsilon > 0$ does not depend on $n$ since the left hand side of the first inequality does not depend on $n$. It also does not depend on $t$ by (c). The term $1/(1+3\lambda)$ is the probability that the particle that was at the center of the star $S$ at the initial time $(n-1)\,C\log t$ dies before infecting any neighboring vertex. The other part comes from the fact that discarding the origin of the star, we can compare the process with three independent processes in $\mathbb{Z}^+$, starting with $\mathbb{Z}^+$ full of particles and then further compare each one of these with a process in $\mathbb{Z}$.

Using the strong Markov property at the moment $T'_{2t}$ when the process $(\xi^S_{S;\,s})_{s\geqslant 0}$ hits $H_{2t}$ and (b) and (c), we obtain:

$$\begin{aligned}P(T\geqslant t) &\leqslant P(T'_{2t}\geqslant t)+P(T'_{2t}<T,\, T'_{2t}<t)\\ &\leqslant P(A_1^c\cap A_2^c\cap \cdots \cap A^c_{\lfloor t/C\log t\rfloor})+c_5\exp\{-c_6 t\}\\ &\leqslant (1-\varepsilon)^{\{t/C\log t\}-1}+c_5\exp\{-c_6 t\}\\ &\leqslant c_3\exp\{-c_4 t/\log t\}\end{aligned}$$

Note now that if $d(y,0)<t$, then our thesis follows immediately from (4.4). Otherwise, using self-duality and thinking of $\mathbb{Z}^+$ as a subgraph of $S$,

$$\begin{aligned}P(\xi^S_{S,\,t}(y)=1) &\leqslant P((\xi^y_{\mathbb{Z}^+;\,s})_{s\geqslant 0}\text{ reaches }0)+P(\xi^y_{\mathbb{Z}^+;\,t}\neq\varnothing)\\ &\leqslant c_5\exp(-c_6 d(y,0))+c_7\exp(-c_8 t)\\ &\leqslant c_9\exp(-c_{10} t)\end{aligned}$$

where we used (b) and (a). ▮

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